An Exact Analysis of Precautionary Consumption Growth

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Abstract

While macro models increasingly incorporate substantial risk, the theoretical knowledge about the effect of uncertainty on consumption growth consists of intuitions from the second order log-linearized Euler equation. I show that the derivation of the log-linearized Euler equation is flawed in that it does not consist in linearizing an Euler equation but in linearizing an ad-hoc mathematical identity. I prove exactly that uncertainty raises consumption growth and makes consumption depart from a random walk. I also prove that this precautionary consumption growth is decreasing in assets, and in transitory and permanent income when income is a transitory-permanent process.

Key words: Precautionary behavior, log-linearized Euler equation, random walk hypothesis

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Introduction

How does uncertainty affects consumption growth and how does its effect varies with the wealth and income of the households? Having exact predictions about the effect of uncertainty on consumption growth and having more predictions about the heterogeneity in the effect of uncertainty on consumption level and growth across wealth and income is getting more valuable as numerical simulations of models that incorporate substantial uncertainty through idiosyncratic shocks have scored a number of empirical successes (see **KaplanViolante2018** for a review of this literature), and as empirical evidence of substantial uncertainty about consumption is piling up (see **NakamuraSergeyevSteinsson2017** for instance).

Yet, what we know about the effect of uncertainty on consumption growth is still based on intuitions from the second order log-linearized Euler equation. Indeed, although the effect of uncertainty on the level of consumption is examined analytically in the literature, with an exact proof that everything else equal uncertainty raises saving and decreases consumption, the effect of uncertainty on consumption growth is only analyzed through the second order log-linearized Euler equation. This is surprisingly true both in the papers that rely on this effect, and in the consumption textbooks (**Deaton1992**, **Attanasio1999**, **JappelliPistaferri2017**). This second order equation expresses log-consumption as a random walk, plus a variance term that is traditionally seen as capturing the effect of precautionary behavior.

In this paper, I help filling this gap: I show that relying on the second order log-linearized Euler equation to get intuitions about the effect of uncertainty on log-consumption growth is problematic because this expression is not based on a linearization of the consumption solution implied by a Euler equation but on the linearization of an ad-hoc identity; I then develop an exact analysis of the contribution of precautionary behavior to consumption growth, and I prove that this contribution is decreasing in assets and in permanent income.

First, the log-linearized Euler equation is derived following a similar method as that developed in the seminal paper of **Hall1978**, which finds that consumption approximately evolves as a random walk. I explain that this random walk expression is obtained by taking the approximation of a mathematical identity, and not the approximation of the solution for consumption implied by the first order condition of the household's maximization problem (the Euler equation). The first and second order version of the log-linearized Euler equation are based on the same method, so they are not linearizations of the Euler equation either. Although the second order log-linearized expression can alternatively be obtained by assuming that future log-consumption is normally distributed, this requires assumptions about the income process that are endogenous to current consumption without the household anticipating this effect of its current consumption on the income process it faces.

I then provide an exact analysis of the effect of precautionary behavior on consumption growth, and I prove three results. The first result, which happens to be similar to the intuition initially provided by the second order log-linearized Euler equation, is that the presence of uncertainty raises consumption growth and does so by an amount that is not deterministic, inducing a departure from a random walk. This result appears true in the second order log-linearized expression, because the variance term that is interpreted as precautionary behavior is positive, so it raises log-consumption growth, and because this variance depends on current variables, so it induces a departure from a random walk. However, in this expression, precautionary consumption growth appears to be second order. This result holds exactly true in a life-cycle model with isoelastic preferences because, when marginal utility is convex, the presence of uncertainty raises the expected marginal utility of future consumption, inducing the household to allocate relatively more consumption to the uncertain future than to the certain present, thus raising consumption growth. The increase in consumption growth depends on the distribution of future consumption, which is affected by current variables, so it induces a departure from a random walk. Yet precautionary consumption growth is not second order but first order around the point at where the variance of future income is zero.

The second result is that this effect of uncertainty in consumption growth is decreasing in assets, and in transitory income when income is a transitory-income process. The third result is that this effect is decreasing in permanent income when income is a transitory-permanent process. These also imply that the effect of uncertainty on the level of consumption is decreasing in assets, transitory income, and permanent income. Intuitively, a gain in assets, transitory income or permanent income shifts the distribution of future consumption upwards, which is a region where the convexity of marginal utility is less pronounced when utility is isoelastic, reducing the effect of uncertainty about future consumption on the expected marginal utility of future consumption. In addition, following a gain in assets, transitory income or permanent income, the same shocks to transitory and permanent income lead to smaller variations in consumption, reducing uncertainty about future consumption. As a result, such a gain reduces the need for precautionary consumption growth. This reduction in precautionary consumption growth translates into a reduction in precautionary saving, that is, in the effect of uncertainty on the level of consumption.

1 Approximating an identity

1.1 The life-cycle model

Household's problem At each period t, a household i chooses its current consumption and the distribution of its future consumption as the solution of the following intertemporal optimization problem:

$$\max_{c_{i,t},\dots c_{i,T}} \sum_{s=0}^{T-t} \beta^{t+s} e^{\delta_{t+s} z_{i,t+s}} E_t \left[u(c_{i,t+s}) \right] \tag{1.1}$$

$$s.t.\quad a_{i,t+k+1} = (1+r)a_{i,t+k} - c_{i,t+k} + y_{i,t+k} \qquad \forall\, 0 \leq k \leq T-t, \tag{1.2}$$

$$c_{i,t+k} > 0, \qquad \forall\, 0 \leq k \leq T-t, \tag{1.3}$$

$$a_{i,T} \geq 0. \tag{1.4}$$

The household is finite-lived with T the length of its life. It has time-separable utility, and at each period t it derives utility from its contemporaneous consumption expenditures $c_{i,t}$. The period utility function $u(c)$ is isoelastic so its functional form is $u(c) = \frac{c^{1-\rho}-1}{1-\rho}$. Future utility is discounted by the factor β, and shifted by the demographic characteristics $z_{i,t}$, whose current and future values are known in advance with certainty by the household. The impact of the demographics on utility is captured by the vector of coefficients δ_t, which can change with calendar time. At each period, the household earns the stochastic amount $y_{i,t}$. The budget constraints (1.2) state that, to store its wealth from one period to another, the household only has access to a risk-free asset that delivers the certain interest rate r, where $a_{i,t}$ denotes the level of this asset at the beginning of period t or at the end of period $t-1$. The conditions (1.3) restrict consumption to be strictly positive at each period. The terminal condition on wealth (1.4) states that the household cannot die with a strictly positive level of debt.

First order condition The first order condition of the maximization problem of the household, known as the Euler equation, is as follows:

$$u'(c_{i,t}) = E_t[u'(c_{i,t+1})]R_{i,t,t+1}, \tag{1.5}$$

with $R_{i,t,t+1} = \beta(1+r)e^{\Delta\delta_{t+1}z_{i,t+1}}$ a factor accounting for the deterministic intertemporal substitution motives. The effect of deterministic intertemporal substitution can equivalently be expressed as a weight $R_{i,t,t+1}^{1/\rho}$ on past consumption: $u'(c_{i,t})R_{i,t,t+1}^{-1} = u'(c_{i,t}R_{i,t,t+1}^{1/\rho})$, which is what I do in the remainder of the paper, and I refer to $c_{i,t}R_{i,t,t+1}^{1/\rho}$ simply as current consumption. This first order condition states that an optimizing household chooses its consumption path so that current and future consumption delivers the same expected marginal utility. Although a natural borrowing limit arises from the combination of the budget constraints, the restrictions requiring consumption to be positive, and the terminal condition on wealth, this limit never binds when utility is isoelastic: the household would never put itself in the situation of possibly consuming zero in the future by borrowing more than the lowest possible amount that it could earn in the future, because its marginal utility approaches infinity when its consumption approaches zero.

1.2 Hall (1978)'s approximation

Hall1978 shows that, in a version of the model above with a quadratic utility and without the conditions restricting current and future consumption to be positive, consumption evolves exactly as a random walk: marginal utility is linear in consumption so the equalization of expected marginal utility over time implies the equalization of expected consumption over time. **Hall1978** also claims that consumption approximately evolves as a random walk with a deterministic trend when utility is isoelastic:

$$c_{i,t+1} \approx c_{i,t}R_{i,t,t+1}^{1/\rho} + \underbrace{\frac{u'(c_{i,t+1}) - E_t[u'(c_{i,t+1})]}{E_t[u'(c_{i,t+1})]}}_{\text{orthogonal to var. at } t}. \tag{1.8}$$

I make explicit that this expression is not derived from approximating the value of consumption implied by the first order condition of the household's problem, but from approximating an identity plugged in this first order condition. More precisely, what **Hall1978** does can be decomposed as follows. He starts from the first order condition (1.5), substitutes for $E_t[u'(c_{i,t+1})] = u'(c_{i,t+1}) + (E_t[u'(c_{i,t+1})] - u'(c_{i,t+1}))$, that is, add $u'(c_{i,t+1}) - u'(c_{i,t+1})$ to the right-hand side, yielding expression (1.6). He then rearranges, and applies $(u')^{-1}(.)$ to each side, yielding expression (1.7). He finally takes a first order approximation of $c_{i,t+1}$ around the point where $u'(c_{i,t+1}) = u'(c_{i,t}R_{i,t,t+1}^{1/\rho})$ (i.e. using $f(x) \approx f(x_0) + (x - x_0)f'(x_0)$ with $f(x) = c_{i,t+1} = (u')^{-1}(u'(c_{i,t+1}))$, $x = u'(c_{i,t+1})$, and $x_0 = u'(c_{i,t}R_{i,t,t+1}^{1/\rho})$) to obtain (1.8):[1]

$$u'(c_{i,t}R_{i,t,t+1}^{1/\rho}) = E_t[u'(c_{i,t+1})] \tag{1.5}$$

$$u'(c_{i,t}R_{i,t,t+1}^{1/\rho}) = u'(c_{i,t+1}) + (E_t[u'(c_{i,t+1})] - u'(c_{i,t+1})) \tag{1.6}$$

$$c_{i,t+1} = (u')^{-1}(u'(c_{i,t+1}) + \cancel{E_t[u'(c_{i,t+1})]} - \cancel{u'(c_{i,t}R_{i,t,t+1}^{1/\rho})}) \tag{1.7}$$

$$c_{i,t+1} \approx c_{i,t}R_{i,t,t+1}^{1/\rho} + \underbrace{\frac{u'(c_{i,t+1}) - u'(c_{i,t}R_{i,t,t+1}^{1/\rho})}{u''(c_{i,t}R_{i,t,t+1}^{1/\rho})}}_{\substack{\text{orthogonal to var. at } t \\ \text{when } u'(c_{i,t}R_{i,t,t+1}^{1/\rho}) = E_t[u'(c_{i,t+1})]}} . \tag{1.8}$$

This means that Hall takes an approximation of $c_{i,t+1}$, a term that he has added to the first order condition, around the point at which $u'(c_{i,t+1})$, also a term that he has added, is close to $u'(c_{i,t}R_{i,t,t+1}^{1/\rho})$. Therefore, the same expression would be obtained by taking a first order approximation of the identity $c_{i,t+1} = (u')^{-1}(u'(c_{i,t+1}))$ around the point at which $u'(c_{i,t+1}) = u'(c_{i,t}R_{i,t,t+1}^{1/\rho})$ (I also make it visible in (1.7) by crossing out the terms that cancel out in the expression

[1]Precisely, Hall defines $\varepsilon_{i,t+1} = u'(c_{i,t+1}) - u'(c_{i,t}R_{i,t,t+1}^{1/\rho})$ as the difference between the marginal utility of current and future consumption and takes a first order approximation around the point where $\varepsilon_{i,t+1} = 0$ (his 'Proof of Corollary 5', second point in his Appendix, p987).

of consumption that is approximated by Hall):

$$c_{i,t+1} = (u')^{-1}(u'(c_{i,t+1})) \tag{1.9}$$

$$c_{i,t+1} \approx c_{i,t}R_{i,t,t+1}^{1/\rho} + \underbrace{\frac{u'(c_{i,t+1}) - u'(c_{i,t}R_{i,t,t+1}^{1/\rho})}{u''(c_{i,t}R_{i,t,t+1}^{1/\rho})}}_{\substack{\text{orthogonal to var. at } t \\ \text{when } u'(c_{i,t}R_{i,t,t+1}^{1/\rho}) = E_t[u'(c_{i,t+1})]}} . \tag{1.10}$$

The Euler equation is of use, not to derive the expressions, but only to have the first order term of the approximation be orthogonal to current shocks: $u'(c_{i,t+1}) - u'(c_{i,t}R_{i,t,t+1}^{1/\rho}) = u'(c_{i,t+1}) - E_t[u'(c_{i,t+1})]$. A differently approximated identity would yield a different expression, in which plugging the Euler equation would not imply that the first order term is orthogonal to current shocks: if instead one were to approximate $c_{i,t+1} = v^{-1}(v(c_{i,t+1}))$, with $v(.)$ a function such that $E_t[v(c_{i,t+1})] \neq v(c_{i,t}R_{i,t,t+1}^{1/\rho})$, around $v(c_{i,t+1}) = v(c_{i,t}R_{i,t,t+1}^{1/\rho})$, thus at the same point as **Hall1978** such that $c_{i,t+1} = c_{i,t}R_{i,t,t+1}^{1/\rho}$, the resulting expression would not be a random walk, although the Euler equation would hold.[2] The random walk expression thus reflects the arbitrary choice of the identity that is linearized.

1.3 The log-linearized Euler equation

Derived with an approximation The first order and second order versions of the log-linearized Euler equation are as follows:

$$ln(c_{i,t+1}) \approx ln(c_{i,t}) + \frac{1}{\rho}ln((1+r)\beta) + \frac{1}{\rho}\Delta\delta_{t+1}z_{i,t+1} + \underbrace{\zeta_{i,t+1}}_{\substack{\text{orthogonal} \\ \text{to var. at } t}} \tag{1.11}$$

$$ln(c_{i,t+1}) \approx ln(c_{i,t}) + \frac{1}{\rho}ln((1+r)\beta) + \frac{1}{\rho}\Delta\delta_{t+1}z_{i,t+1} + \underbrace{E_t[\varepsilon_{i,t+1}^2]}_{\text{precaution}} + \underbrace{\zeta_{i,t+1}}_{\substack{\text{orthogonal} \\ \text{to var. at } t}} , \tag{1.12}$$

[2] The resulting expression would be $c_{i,t+1} \approx c_{i,t}R_{i,t,t+1}^{1/\rho} + \frac{v(c_{i,t+1}) - v(c_{i,t}R_{i,t,t+1}^{1/\rho})}{v''(c_{i,t}R_{i,t,t+1}^{1/\rho})}$ with $v(c_{i,t}R_{i,t,t+1}^{1/\rho}) \neq E_t[v(c_{i,t+1})]$.

with $\varepsilon_{i,t+1} = \frac{u'(c_{i,t+1}) - E_t[u'(c_{i,t+1})]}{E_t[u'(c_{i,t+1})]}$ the relative innovation to the marginal utility of consumption between t and $t+1$, and $\zeta_{i,t+1} = ln(c_{i,t+1}) - E_t[ln(c_{i,t+1})]$ the innovation to log-consumption between t and $t+1$. I clarify that their derivations rely on the same principle as **Hall1978**: start from the Euler equation, add $u'(c_{i,t+1})$ on each side by substituting $E_t[u'(c_{i,t+1})]$ with $u'(c_{i,t+1}) + (E_t[u'(c_{i,t+1})] - u'(c_{i,t+1}))$, and rearrange to have log-consumption on one side and a shock to the marginal utility of consumption on the other, which are both variables that are not initially in the Euler equation:

$$ln(c_{i,t+1}) = ln(c_{i,t}) + \frac{1}{\rho} ln(R_{i,t,t+1}) - \frac{1}{\rho} ln(1 + \underbrace{\frac{u'(c_{i,t+1}) - u'(c_{i,t} R_{i,t,t+1}^{1/\rho})}{u'(c_{i,t} R_{i,t,t+1}^{1/\rho})}}_{\varepsilon_{i,t+1}}) \tag{1.13}$$

Expressions (1.11) and (1.12) are obtained as first and second order approximations of (1.13) around the point where $\varepsilon_{i,t+1} = \frac{u'(c_{i,t+1}) - u'(c_{i,t} R_{i,t,t+1}^{1/\rho})}{u'(c_{i,t} R_{i,t,t+1}^{1/\rho})} = 0$. Again the expression (1.13) is in fact the identity $ln(c_{i,t+1}) = -\frac{1}{\rho} ln(u'(c_{i,t+1}))$, and the expressions (1.11) and (1.12) can equivalently be obtained as first and second order approximations of this identity around the point where $u'(c_{i,t+1}) = u'(c_{i,t} R_{i,t,t+1}^{1/\rho})$. Although the second order version of the log-linearized Euler equation is not a random walk expression, it still relies on the approximation of an arbitrary identity.

Derived in the special case of log-normal consumption As noted by **Deaton1992**, it is also possible to obtain the second order log-linearized Euler equation by assuming that the expected distribution of future log-consumption is Gaussian.[3] The problem with this assumption is that it requires having hypotheses about the distribution of the income process that are endogenous to the value of current consumption (which influences the distribution of future log-consumption), without the household internalizing the effect of its consumption decision on the

[3] See **Deaton1992** p.64: rearranging the first order condition yields $E_t[exp(-\rho(\Delta ln(c_{i,t+1}) + \frac{1}{\rho} ln(R_{i,t,t+1}))]$ (or $E_t[exp(-\rho(\Delta ln(c_{i,t+1}) + \frac{1}{\rho}(R_{i,t,t+1} - 1))]$ if additionally taking a first order approximation around $R_{i,t,t+1} = 1$). Gaussianity of $ln(c_{i,t+1})$ combined with the fact that $R_{i,t,t+1}$ is deterministic and $ln(c_{i,t})$ known at t implies that $(\Delta ln(c_{i,t+1}) + \frac{1}{\rho} ln(R_{i,t,t+1}))$ is Gaussian. It writes $E_t[\Delta ln(c_{i,t+1})] = \frac{1}{\rho} ln(R_{i,t,t+1}) + 12\rho\omega_{i,t}^2$, with $\omega_{i,t} = var_t(ln(c_{i,t+1}))$.

income process that it faces. To see this, note for instance that at the last period of life, $t = T$, consumption is $ln(c_{i,T}) = ln((1+r)a_{i,T} + y_{i,T})$, with $(1+r)a_{i,T}$ constant from the point of view of period $T-1$. Making the expected distribution of $ln(c_{i,T})$ at $T-1$ normal is possible but requires assumptions about the distribution of $y_{i,T}$ that depend on the value of $(1+r)a_{i,T}$, thus on the value of $c_{i,T-1}$.

1.4 The log-linearized Euler equation

From my discussion of Hall's method, taking an approximation of the consumption implied by the first order condition (1.5) around small innovations between t and $t+1$ is impossible because there are no variables realized at $t+1$ in (1.5)—except $R_{i,t,t+1}$ but it is deterministic and thus similarly independent of the innovations between t and $t+1$. Doing so anyway by adding functions of $c_{i,t+1}$ on each side of (1.5) leads to taking the approximation of an identity.

Yet, another erroneous way of deriving a random walk expression of consumption is by taking simultaneous approximations of $u'(c_{i,t+1})$ around $c_{i,t+1} = c_{i,t}R_{i,t,t+1}^{1/\rho}$ in each possible state of the world at $t+1$, regardless of the income shock that realizes, applying an expectation operator, and putting the expression in the first order condition. In the discrete case with J possible future states of the world, and each state $j \in J$ occurring with probability π_j, such an approximation consists in:

$$
\begin{aligned}
u'(c_{i,t}R_{i,t,t+1}^{1/\rho}) &= \sum_{j\in J} \pi_j u'(c_{i,t+1}^j) \\
u'(c_{i,t}R_{i,t,t+1}^{1/\rho}) &\approx \sum_{j\in J} \pi_j u'(c_{i,t}R_{i,t,t+1}^{1/\rho}) + \sum_{j\in J} \pi_j (c_{i,t+1}^j - c_{i,t}R_{i,t,t+1}^{1/\rho}) u''(c_{i,t}R_{i,t,t+1}^{1/\rho}) \\
0 &\approx E_t[c_{i,t+1}] - c_{i,t}R_{i,t,t+1}^{1/\rho}.
\end{aligned}
$$

Yet, it is a way of doing that obliterates the effect of uncertainty about $c_{i,t+1}$ on the decision about $c_{i,t}$: $u'(c_{i,t+1})$ is approximated taking $c_{i,t}$ as realized, and the expression of $E_{i,t}[u'(c_{i,t+1})]$ obtained is then plugged in the first order condition $u'(c_{i,t}) = E_{i,t}[u'(c_{i,t+1})]$ that determines the value of $c_{i,t}$.

2 Exact analysis

2.1 Precautionary consumption growth and precautionary saving

Precautionary consumption growth In the first order condition of the household's maximization problem, when marginal utility is decreasing and convex, the presence of uncertainty about future consumption raises the solution for expected consumption growth above the value it would take in the absence of uncertainty:

$$u'(c_{i,t}R_{i,t,t+1}^{1/\rho}) = E_t[u'(c_{i,t+1})] = u'(E_t[c_{i,t+1}] - \varphi_{i,t}), \tag{2.1}$$

$$E_t[c_{i,t+1}] = c_{i,t}R_{i,t,t+1}^{1/\rho} + \underbrace{\varphi_{i,t}}_{\substack{>0 \text{ when } u'''>0 \\ \text{and } -u''>0}}. \tag{2.2}$$

Indeed, when marginal utility is convex ($u''' > 0$), the effects of negative and positive innovations to consumption are asymmetric: an income shock that reduces consumption below its expected value increases the marginal utility of consumption more than an income shock that raises consumption above its expected value by the same amount decreases its marginal utility. Thus, the presence of innovations to future consumption increases the expected marginal utility of consumption above the marginal utility of expected consumption: $E_t[u'(c_{i,t+1})] > u'(E_t[c_{i,t+1}])$. When marginal utility is decreasing ($-u'' > 0$), it induces the household to set its consumption at t below its expected consumption at $t+1$, by the amount $\varphi_{i,t} > 0$ such that: $E_t[u'(c_{i,t+1})] = u'(E_t[c_{i,t+1}] - \varphi_{i,t})$.[4] I refer to $\varphi_{i,t}$ as precautionary consumption growth because it captures the effect of uncertainty on consumption growth. Precautionary consumption growth $\varphi_{i,t}$ is not in general deterministic, because it depends on the expected distribution at t of $c_{i,t+1}$. Conditional on age, the model (1.1)-(1.4) implies that this distribution is entirely determined by the current level of assets, current income, and current expected distribution future income:

$$\varphi_{i,t} = g_t(a_{i,t}, y_{i,t}, f_{y_{i,t+1}}, \ldots, f_{y_{i,T}}). \tag{2.3}$$

[4] This is strictly positive when marginal utility is decreasing and convex by Jensen's inequality.

Thus, observing the level of assets, or income, at t can help predict the value of $\varphi_{i,t} = E_t[c_{i,t+1}] - c_{i,t}R_{i,t,t+1}^{1/\rho}$, and consumption growth covaries with past variables: it does not evolve as a random walk with a deterministic trend.

Precautionary saving Iterating forward on equation (2.2), consumption growth between t and any future period $t+s$ is a weighted sum of the precautionary premiums between these two dates: $E_t[c_{t+s}] = c_t R_{t,t+s}^{1/\rho} + \sum_{k=1}^{s} E_t[\varphi_{t+k-1}] R_{t+k,t+s}^{1/\rho}$. I plug these expressions into the intertemporal budget constraint that arises from the combination of (1.2)-(1.4), stating that the net present value of current and future expected consumption equals current assets plus the net present value of current and future expected income, and rearranging, to obtain the following equilibrium relationship satisfied by the consumption solution of the life-cycle model:

$$c_t = \underbrace{\frac{1}{l_{t,0}}\left((1+r)a_t + \sum_{s=0}^{T-t}\frac{E_t[y_{t+s}]}{(1+r)^s}\right)}_{\substack{\text{consumption under perfect foresight} \\ \frac{1}{l_{t,0}}W_{i,t}}} - \underbrace{\frac{1}{l_{t,0}}\left(\sum_{s=1}^{T-t}\frac{l_{t,s}E_t[\varphi_{t+s-1}]}{(1+r)^s}\right)}_{\substack{\text{precautionary saving} \\ PS_{i,t}}},$$

with $\frac{1}{l_{t,k}} = \left(\sum_{s=k}^{T-t}\frac{R_{t,t+s}^{1/\rho}}{(1+r)^s}\right)^{-1}$ the weight put on consumption at period t relative to consumption between t and $T-k$.[5] This expression encompasses the perfect foresight case, in which the household simply consumes a fraction $\frac{1}{l_{t,0}}$ of its total expected lifetime resources—the sum of its net assets, current income and expected future income—denoted W_t (for wealth). In the presence of uncertainty, to be able to implement the additional precautionary consumption growth it desires, the household takes out its total expected precautionary growth from its total expected resources and consumes a constant share of what remains. As precautionary saving is defined as the difference between actual consumption and the consumption that would be chosen under perfect foresight everything else being equal, it writes as a weighted sum of the current and future expected value of precautionary consumption growth between two consecutive periods φ.

Robustness to approximations First order approximations of $u'(c_{i,t}) = E_{i,t}[u'(c_{i,t+1})]$

[5]When consumers are neither patient nor impatient ($\beta = \frac{1}{1+r}$) and individual characteristics are constant ($z_t = z$), $l_{t,0}$ tends toward $\frac{r}{1+r}$ as T approaches infinity.

around the points where the variance of future income is zero, $var_t(y_{i,t+1}) = 0$, would not yield random walk expressions because the first order term of such an approximation covaries with past variables in general:

$$E_t[c_{i,t+1}] - c_{i,t}R^{1/\rho}_{i,t,t+1} = \varphi_{i,t} \approx 0 + var_t(y_{i,t+1}) \underbrace{\frac{dc_{i,t}}{dvar_t(y_{i,t+1})}}_{\substack{\text{correlates} \\ \text{with var. at } t}} \tag{2.4}$$

Also, approximations of the consumption implied by the first order condition (1.5) around innovations at t (i.e. around the point where for a variable x, $x_{i,t} = E_{t-1}[x_{i,t}]$) are possible, because innovations at t are present in (1.5), but except for possibly knife-edge realizations of $x_{i,t}$, they do not yield expressions of $E_t[c_{i,t+1}] - c_{i,t}R^{1/\rho}_{i,t,t+1}$ that are zero.

Discussion In the existing literature, this result that uncertainty raises consumption growth is acknowledged but obtained from the second order log-linearized Euler equation, rather than from an exact rearrangement of the first order condition as I do with equation (2.2): the literature reviews such as those of **BrowningLusardi1996** and **AttanasioWeber2010**, as well as textbooks such as **Deaton1992** (Chapter 2.2 p64, and Chapter 6.1 p179-180 'An Approximation and a Special Case'), **Attanasio1999** (p768), and **JappelliPistaferri2017** (p102), present the result that uncertainty increases consumption growth as an outcome of the second order log-linearized Euler equation. The further result that precautionary behavior induces a departure from a random walk is present in the literature but also derived from the second order log-linearized Euler equation. As noted by **Deaton1992**, **Carroll1997** is one paper that remarks that the variance term in this second order expression induces a departure from a random walk. **Carroll2001** later makes more forcefully the point that the variance term should not be omitted.

In parallel, the literature on precautionary behavior acknowledges that the presence of uncertainty increases the expected marginal utility of future consumption in a standard life-cycle model, but the mechanism is invoked to analyze the effect of uncertainty on the level of consumption and not on the growth of consumption.[6]

[6]**Deaton1992** notes that the convexity of the marginal utility of consumption raises its expected value in the presence of uncertainty (Chapter 1.3 p.29, and Chapter 6.1 p.177-178), but

Finally, the precautionary consumption growth that I derive is similar but not identical to the precautionary premium of **Kimball1990b**: Kimball's precautionary premium is a comparison of marginal utilities across states (certain versus uncertain) applied to a two-period model (in which uncertainty about future consumption is exogenous and coincides with uncertainty about future income); precautionary consumption growth is a comparison across time (current versus future), which implies a comparison across states (current is certain and future is uncertain), in a multiperiod model (thus in which the uncertainty about future consumption is not fixed but endogenous).[7]

2.2 Variations with assets and income

Income process Log-income is a transitory-permanent process.[8] It writes as the sum of a permanent component that evolves as a random walk, and of a transitory component that evolves as an $MA(q)$:

$$ln(y_{i,t}) = p_{i,t} + \mu_{i,t} \tag{2.5}$$
$$with \begin{cases} p_{i,t} &= p_{i,t-1} + \eta_{i,t} \\ \mu_{i,t} &= \varepsilon_{i,t} + \theta_1 \varepsilon_{i,t-1} + ... + \theta_q \varepsilon_{i,t-q}. \end{cases}$$

The uncertainty of the household about its future income comes from the presence of the shocks, $\eta_{i,t}$ and $\varepsilon_{i,t}$. The shock $\eta_{i,t}$ is a permanent shock because

then applies it directly to the study of the level of consumption instead of consumption growth. To do so, the shock he considers is an exogenous increase in the variance of future consumption, keeping the expected value of future consumption constant. Such a shock is inconsistent with the model, however, because a decrease in current consumption increases the resources available for future consumption, thus increase the expected value of future consumption, which cannot be kept constant. **Attanasio1999** indicates that the effect of uncertainty on expected consumption growth is the cause of the eventual decrease in current consumption (p770), but still bases the finding that log-consumption growth increases with uncertainty on a log-linearized equation and not on the general model.

[7] In particular, the difference makes it impossible to directly use Kimball's result that the precautionary premium is decreasing in assets when utility is isoelastic, and this is why I only prove the result that precautionary consumption growth decreases with assets under some conditions on the distribution of future income.

[8] It is not necessary to specify an income process to show that existing random walk expressions are not based on the first order condition of the household problem, and that there is a precautionary component to consumption growth that does not disappear in first order approximations around small innovations. Yet, to show that this precautionary component of consumption growth correlates with past variables, and in particular past transitory shocks, it is simpler to have one.

its realization remains in the value of p, so it modifies the income received by the household at all following periods. The shock $\varepsilon_{i,t}$ is transitory because its effect on income dissipates after q periods.[9] At each period, the permanent and transitory shocks are drawn independently of each other and independently of their previous realizations. Their variances are denoted $\sigma^{\eta}_{i,t}$ and $\sigma^{\varepsilon}_{i,t}$.

Theorem I prove that the following three results hold exactly in the model described by (1.1)-(1.4), and when income is described by (2.5):

(i) $\frac{dE_t[c_{i,t+1}]-c_{i,t}R^{1/\rho}_{i,t,t+1}}{da_{i,t}} = \frac{d\varphi_{i,t}}{da_{i,t}} < 0$ and $\frac{dPS_{i,t}}{da_{i,t}} < 0$

(ii) $\frac{dE_t[c_{i,t+1}]-c_{i,t}R^{1/\rho}_{i,t,t+1}}{d\varepsilon_{i,t}} = \frac{d\varphi_{i,t}}{d\varepsilon_{i,t}} < 0$ and $\frac{dPS_{i,t}}{d\varepsilon_{i,t}} < 0$

(iii) $\frac{dE_t[c_{i,t+1}]-c_{i,t}R^{1/\rho}_{i,t,t+1}}{dp_{i,t}} = \frac{d\varphi_{i,t}}{dp_{i,t}} < 0$ and $\frac{dPS_{i,t}}{dp_{i,t}} < 0$

Proof (i) I derive both sides of the first order condition of the maximization problem with respect to a change in assets and divide by $(-u''(c_{i,t}R^{1/\rho}_{i,t,t+1}))$:

$$\frac{dc_{i,t}R^{1/\rho}_{i,t,t+1}}{da_{i,t}} = E_t[\frac{dc_{i,t+1}}{da_{i,t}}\frac{(-u''(c_{i,t+1}))}{(-u''(c_{i,t}R^{1/\rho}_{i,t,t+1}))}] \tag{2.6}$$

$$\frac{dc_{i,t}R^{1/\rho}_{i,t,t+1}}{da_{i,t}} = \frac{dE_t[c_{i,t+1}]}{da_{i,t}} \underbrace{\frac{E_t[-u''(c_{i,t+1})]}{(-u''(c_{i,t}R^{1/\rho}_{i,t,t+1}))}}_{>1} + \underbrace{cov_t\left(\frac{dc_{i,t+1}}{da_{i,t}}, -u''(c_{i,t+1})\right)}_{>0}. \tag{2.7}$$

Two effects induce current consumption to respond more than expected future consumption to a gain in assets. First, it responds more because $E_t[-u''(c_{i,t+1})] > -u''(c_{i,t}R^{1/\rho}_{i,t,t+1})$. This inequality holds true because, when utility is isoelastic, $-u''(.)$ is a convex function of $u'(.)$.[10] Thus, at the value of current consumption such that $E_t[u'(c_{i,t+1})] = u'(c_{i,t}R^{1/\rho}_{i,t,t+1})$, one has that $E_t[-u''(c_{i,t+1})] >$

[9] By construction, at the end of the household's life the effect of a transitory shock can last until the last period of the household's life, and resembles a permanent shock.

[10] This is actually true of any function $u(.)$ that displays decreasing absolute prudence, that is $\frac{u'''}{(-u'')}(.) < 0$. Decreasing absolute prudence of $u(.)$ implies that $\frac{u'''}{(-u'')} < \frac{(-u'''')}{u'''}$, thus that the absolute risk aversion of $u'(.)$ is always smaller than the absolute risk aversion of $-u''(.)$. By **Pratt1964-Arrow1965**, this means that $-u''(.)$ is a convex function of $u'(.)$.

$-u''(c_{i,t}R^{1/\rho}_{i,t,t+1})$.[11] Intuitively, the property that $-u''(.)$ is a convex function of $u'(.)$ means that the convexity of marginal utility is less pronounced around higher levels of consumption.[12] Thus, a gain in assets that increases future consumption moves its distribution to a region where the convexity is less pronounced thus where the effect of uncertainty on the expected marginal utility of future consumption is smaller, which is why the expected marginal utility of future consumption falls more than the marginal utility of current consumption with a shift upwards in consumption.

The covariance $cov_t\left(\frac{dc_{i,t+1}}{da_{i,t}}, -u''(c_{i,t+1})\right)$ is positive, because both terms are decreasing in the value of the transitory shock $\varepsilon_{i,t+1}$, which is the only source of covariance from the perspective of period t since there are no permanent shocks in the constrained model. Indeed, I show in **Commault2019** that the cross partial derivative of consumption with respect to assets and a transitory income shock and with respect to assets and a permanent income shock are both negative. Thus, the response of consumption at $t+1$ to a gain in assets at t, and therefore a gain in assets at $t+1$, is decreasing in $\varepsilon_{i,t+1}$ and $\eta_{i,t+1}$: $\frac{dc_{i,t+1}}{da_{i,t}\varepsilon_{i,t+1}} < 0$ and $\frac{dc_{i,t+1}}{da_{i,t}\eta_{i,t+1}} < 0$. Now, $c_{i,t+1}$ is increasing in $\varepsilon_{i,t+1}$ and $\eta_{i,t+1}$, so the absolute value of the slope of marginal utility $-u''(c_{i,t+1})$ is decreasing in $\varepsilon_{i,t+1}$ and $\eta_{i,t+1}$: $\frac{d-u''(c_{i,t+1})}{d\varepsilon_{i,t+1}} = \frac{dc_{i,t+1}}{d\varepsilon_{i,t+1}}(-u'''(c_{i,t+1})) < 0$ and $\frac{d-u''(c_{i,t+1})}{d\eta_{i,t+1}} = \frac{dc_{i,t+1}}{d\eta_{i,t+1}}(-u'''(c_{i,t+1})) < 0$. Because both terms in the covariance are decreasing in $\varepsilon_{i,t+1}$ and in $\eta_{i,t+1}$, which are the only sources of uncertainty between t and $t+1$, the two terms covary positively: $cov_t\left(\frac{dc_{i,t+1}}{da_{i,t}}, -u''(c_{i,t+1})\right) > 0$. Intuitively, when a gain in assets increases future consumption most in the states of the world in which it is the lowest, it reduces the dispersion of future consumption thus the uncertainty about future consumption, and eventually the effect of this uncertainty on the expected marginal utility of future consumption. These two effects imply that:

$$\frac{dE_t[c_{i,t+1}]}{da_{i,t}} - \frac{dc_{i,t}R^{1/\rho}_{i,t,t+1}}{da_{i,t}} = \frac{d\varphi_{i,t}}{da_{i,t}} =< 0.$$

[11]More precisely, denoting $\kappa_{i,t}$ the amount such that $E_t[-u''(c_{i,t+1})] = -u''(E_t[c_{i,t+1}] - \kappa_{i,t})$ and $\varphi_{i,t}$ the amount such that $E_t[u'(c_{i,t+1})] = u'(E_t[c_{i,t+1}] - \varphi_{i,t})$, the fact that $-u''(.)$ is a convex function of $u'(.)$ implies that $\kappa_{i,t} > \varphi_{i,t}$ (**Pratt1964-Arrow1965**). Because $-u''(.)$ is decreasing, $E_t[-u''(c_{i,t+1})] = -u''(E_t[c_{i,t+1}] - \kappa_{i,t}) > -u''(E_t[c_{i,t+1}] - \varphi_{i,t}) = -u''(c_{i,t}R^{1/\rho}_{i,t,t+1})$.

[12]Formally, a small increase in consumption Δc, is equivalent to a change in marginal utility from $u'(.)$ to $u'(.) + \Delta c u''(.)$, which corresponds to decrease in the convexity of marginal utility, because $u'(.)$ is a convex function of $u'(.) + \Delta c u''(.)$.

A change in current assets only affects future expected precautionary consumption growth through its effect on future assets. For any $s > 0$:

$$\frac{dE_t[\varphi_{i,t+s}]}{da_{i,t}} = E_t\left[\underbrace{\frac{da_{i,t+s}]}{da_{i,t}}}_{>0}\underbrace{\frac{d\varphi_{i,t+s}}{da_{i,t+s}}}_{<0}\right] < 0.$$

As a result, precautionary saving is decreasing in assets:

$$\frac{dPS_{i,t}}{da_{i,t}} = \frac{1}{l_{t,0}}\left(\sum_{s=1}^{T-t}\frac{l_{t,s}}{(1+r)^s}\frac{dE_t[\varphi_{t+s-1}]}{da_{i,t}}\right) < 0.$$

Proof (ii) and (iii) Deriving each side of the Euler equation with respect to $\varepsilon_{i,t}$ and $\eta_{i,t}$ yields expressions similar to (2.7). The term $fracE_t[-u''(c_{i,t+1})](-u''(c_{i,t}R^{1/\rho}_{i,t,t+1}))$ is exactly identical. The covariances $cov_t\left(\frac{dc_{i,t+1}}{d\varepsilon_{i,t}}, -u''(c_{i,t+1})\right)$ and $cov_t\left(\frac{dc_{i,t+1}}{d\eta_{i,t}}, -u''(c_{i,t+1})\right)$ are positive as well, because **Commault2019** also proves that consumption is concave in the transitory and permanent shocks, and that the cross partial derivatives of consumption with respect to each type of shock is negative. As a result, the same reasoning as above implies that:

$$\frac{dE_t[c_{i,t+1}]}{d\varepsilon_{i,t}} - \frac{dc_{i,t}R^{1/\rho}_{i,t,t+1}}{d\varepsilon_{i,t}} = \frac{d\varphi_{i,t}}{d\varepsilon_{i,t}} =< 0,$$
$$\frac{dE_t[c_{i,t+1}]}{d\eta_{i,t}} - \frac{dc_{i,t}R^{1/\rho}_{i,t,t+1}}{deta_{i,t}} = \frac{d\varphi_{i,t}}{deta_{i,t}} =< 0.$$

Eventually, it yields that:

$$\frac{dPS_{i,t}}{d\varepsilon_{i,t}} < 0, \frac{dPS_{i,t}}{d\eta_{i,t}} < 0.$$

Discussion Carroll1997 notes that the variance term in the second order Euler equation is decreasing in assets, which would imply result (i), that precautionary behavior makes log-consumption growth decreasing in assets.[13] The two caveats are, first, that the second order Euler equation is based on an identity and does not reflect the first order condition of a life-cycle model; second,

[13] See p12: '...the gap between $E_t[\Delta ln(c_{t+1})]$ and the $\rho^{-1}(r-\delta)$ line [that is the variance term] is strongly declining in the level of gross wealth ratio. This happens for the intuitive reason that consumers with less wealth have less ability to buffer their consumption against shocks to income.'

that the proposition that the variance term is decreasing in wealth holds when households are only subject to transitory shocks, using the concavity result of **CarrollKimball1996**, but not necessarily when they are subject to permanent shocks, as in the paper's model.

3 Conclusion

Contrary to what could have been suggested by the random walk expression of consumption and the log-linearized Euler equation, neither consumption nor log-consumption approximately evolve as random walks around small shocks when consumption is the solution of a standard life-cycle model with uncertainty.

However, the intuitions about the effect of precautionary behavior on consumption growth that were derived from the second order Euler equation mostly hold true in this model: precautionary behavior raises consumption growth, and makes consumption depart from a random walk because its effect is predictable from current and past variables still holds in the general case. One important difference, however, is that the effect of precautionary behavior is not second order around small shocks. In addition, a gain in assets, transitory income, or permanent income reduces precautionary consumption growth and precautionary in a life-cycle model with isoelastic utility.