On the Concavity of Consumption in the Permanent Component of Earnings

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Abstract

The extent to which people consume and save out of their resources matters to understand wealth accumulation. The result that consumption is concave in accumulated wealth has been extremely influential. However, the reason for the concavity relates to the fact that accumulated wealth is safe and reduces precautionary saving. The same result thus might thus not apply to risky human capital, which represents a large share of total expected resources. In this paper, I prove that consumption is also concave in the permanent component of earnings, that scales human capital. The mechanism is the opposite that the one behind the concavity in wealth: at a higher level of permanent earnings, the precautionary saving motive is stronger, and more sensitive to variations in permanent earnings.

1 Intro

The extent to which people consume and save out of their resources, and the reason why they do, matters to understand long-term trends and the impact of structural changes on the economy. One important existing result is that consumption is concave in risk-free accumulated wealth (Carroll and Kimball (1996)): everything else being equal, an increase in wealth raises consumption, but less so at a higher level of wealth. Under the most common utility assumptions, the reason for this concavity is that the precautionary gap between consumption in the presence of uncertainty and consumption absent uncertainty decreases convexly with wealth.

Now, besides accumulated wealth, a major resources for most households is their current and expected future earnings, that is, their human capital. Contrary to risk-free accumulated wealth,

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though, human capital is subject to shocks. It is therefore unclear that an increase in its expected value can reduce convexly precautionary saving.

In this paper, I prove that consumption is concave in the permanent component of earnings, where the permanent component is a scaling factor that multiplies the current and future realizations of earnings. The mechanism here is that the precautionary gap between consumption in the presence of uncertainty and consumption absent uncertainty increases convexly with permanent earnings.

The typical difficulty is that there is no closed-form solution for consumption in this model, that is, there is no expression relating the consumption solution to its exogenous determinants. Existing analytical results examine subcases of the standard model. In those subcases, consumption is linear in the permanent component of income. This lead to a general notion that consumption is probably linear in permanent earnings. The seminal permanent income hypothesis of **Friedman1957** implies that, in the absence of uncertainty, people consume a fixed fraction of their total expected flow of income at each period. With a permanent, multiplicative component to earnings, this means that consumption is linear in the permanent component income. Later on, Carroll (2006) shows that the consumer's maximization problem scales in permanent income, which is not a linearity results but still suggests that the persistent component of income is something that can simply be scaled away. Recently, Straub (2019) examines a subcase in which initial assets are proportional to a fixed-effect component of income, and show that consumption is then proportional to this fixed-effect component as well, which he refers to as 'permanent income'.[1] Straub (2019) also notes that if one is willing to depart from the standard model and assume non-homothetic preferences, consumption becomes concave in this fixed-effect component.

2 A standard income-fluctuation model

I consider a simple income-fluctuation problem with a transitory-permanent earnings process and only one asset. I later discuss extensions of this framework.

A consumer i is finite-lived, with T the length of their life. The consumer chooses consumption expenditures at period t, denoted c_t^i, to maximize lifetime expected utility subject to a number of

[1] His result is about proportionality, which is even stronger than linearity: if the fixed effect component is multiplied by x, consumption is multiplied by x as well.

constraints

$$V_t^i(a_t^i, e^{p_t^i}, e^{\varepsilon_t^i}) = \max_c \; u(c) + \beta E_t \left[V_{t+1}^i(a_{t+1}^i, e^{p_{t+1}^i}, e^{\varepsilon_{t+1}^i}) \right] \tag{2.1}$$

$$\begin{aligned} with \;\; \text{Utility conditions: } & u'(.) > 0, u''(.) < 0, \text{ and } u'''(.) > 0 && (2.2)\\ \text{Positive spending: } & c > 0, && (2.3)\\ \text{Budget constraint: } & a_{t+1}^i = (1+r)a_t^i + y_t^i - c, && (2.4)\\ \text{Earnings: } & y_t^i = e^{p_t^i} e^{\varepsilon_t^i} && (2.5)\\ \text{Permanent component: } & e^{p_{t+1}^i} = e^{p_t^i} e^{\eta_{t+1}^i}, && (2.6)\\ \text{Terminal wealth: } & a_{T+1}^i \geq 0. && (2.7)\end{aligned}$$

Utility is time-separable and at each period depends only on contemporaneous consumption. The period utility function $u(.)$ is such that marginal utility is positive, decreasing, and convex in consumption: $u'(.) > 0$, $u''(.) < 0$, and $u'''(.) > 0$. This implies that people are prudent, so uncertainty pushes them to save more than they would have otherwise. The marginal utility $u'(c)$ approaches infinity when consumption c approaches zero. The discount factor β captures how much consumers discount utility between two consecutive periods.

The positive consumption condition (2.3) imposes that consumption be strictly positive at each period.

The budget constraint (2.4) states that, to store their wealth from one period to the next the consumer only has access to one risk-free liquid asset. The term a_t^i denotes the level of this asset at the beginning of period t—or at the end of $t-1$. The risk-free return rate is r. This rate r is such that $\beta(1+r) \leq 1$.

The labor earnings specification, described with (2.5) and (2.6), means that earnings are a transitory-permanent process: earnings are the product of a permanent component $e^{p_t^i}$ that evolves as a multiplicative random walk and of a transitory innovation $e^{\varepsilon_t^i}$ that is an i.i.d. shock. Because the permanent component $e^{p_t^i}$ multiplies the value of the permanent component at the next period, it multiplies each realization of earnings until the rest of the consumer's lifetime: at $t+s$, earnings are $y_{t+s}^i = e^{p_t^i} e^{\eta_{t+1}^i + \ldots + \eta_{t+s}^i} e^{\varepsilon_t^i}$. It thus plays the role of a scaling factor. Note that this specification encompasses an even simpler specification in which the permanent component is just a multiplicative fixed effect $e^{p_t^i} = e^{p^i}$. This is for instance the specification in Straub (2019). Incidentally, the transitory-permanent process has initially been used to model the earnings of individuals (e.g. in Meghir and Pistaferri (2004)) but is now used more broadly to model the net income of households, including the effect of taxes and transfers (e.g. in Blundell, Pistaferri, and Preston (2008) or in numerical simulations). In this theoretical part, I assume for simplicity that earnings and net income coincide—there are no taxes nor transfers. In the empirical and numerical part, the transitory-permanent process models earnings. For the precautionary motive to be strictly positive, I impose that people face a strictly positive amount of transitory earnings uncertainty: $var(\varepsilon) > 0$.

The terminal condition on wealth (2.7) states that the consumer cannot die with a strictly positive level of debt: assets at the end of the last period T—and the beginning of $T+1$—have to be non-negative. The combination of this condition with the period budget constraints and positive spending constraints generates a natural borrowing constraint that prevents people from holding a level of debt superior to what they could ever repay. This constraint never binds because marginal utility approaches infinity as consumption approaches zero: consumers would never put themselves in the situation of possibly consuming zero in the future. In the remainder of the section, I drop the household index i to ease notations.

3 The effect of the permanent component on consumption

Theorem. In the model described above by (2.1)-(2.7), at any period $t < T-1$, when the ratios of temperance over prudence and prudence over risk-aversion are both non-increasing, consumption is strictly concave in wealth. This means that the response of consumption to a change in permanent earnings is lower at a higher level of permanent earnings e^{p_t}

$$\frac{\partial^2 c_t}{\partial (e^{p_t})^2} < 0$$

Proof of the Theorem: In order to prove the Theorem, I first prove a Lemma stating that, in the model above, at any period $t < T-1$, $\frac{\partial^2 c_{t+1}}{\partial a_{t+1} \partial e^{p_{t+1}}} < \sqrt{\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2} \frac{\partial^2 c_{t+1}}{\partial (e^{p_{t+1}})^2}}$. I prove it in the Online Appendix A. I then prove the Theorem by backward induction. At the last period $t = T$, consumers consume all of their remaining resources $c_T = (1+r)a_T + e^{\varepsilon_T} e^{p_T}$. Consumption is linear in e^{p_T}

$$\frac{\partial^2 c_T}{\partial (e^{p_T})^2} = \frac{\partial e^{\varepsilon_T}}{\partial e^{p_T}} = 0. \tag{3.1}$$

The Theorem therefore holds true, not strictly, at the last period. I assume that it holds true not strictly at $t+1$ and show that it must then hold true strictly at t. I derive each side of the Euler equation with respect to a change in e^{p_t}:

$$\frac{\partial c_t}{\partial e^{p_t}}(-u''(c_t)) = E_t[\frac{\partial c_{t+1}}{\partial e^{p_t}}(-u''(c_{t+1}))] \tag{3.2}$$

$$\frac{\partial c_t}{\partial e^{p_t}}(-u''(c_t)) = E_t[\big(\frac{\partial a_{t+1}}{\partial e^{p_t}}\frac{\partial c_{t+1}}{\partial a_{t+1}} + \frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial c_{t+1}}{\partial e^{p_{t+1}}}\big)(-u''(c_{t+1}))] \tag{3.3}$$

I derive each side a second time with respect to e^{p_t} and rearrange, substituting $\frac{\partial^2 a_{t+1}}{\partial (e^{p_t})^2} = -\frac{\partial^2 c_t}{\partial (e^{p_t})^2}$ and $\frac{\partial c_t}{\partial e^{p_t}} = E_t[\frac{\partial c_{t+1}}{\partial e^{p_t}}\frac{-u''(c_{t+1})}{-u''(c_t)}]$ and $\frac{\partial^2 e^{p_{t+1}}}{\partial (e^{p_t})^2} = 0$

$$\begin{aligned}
\frac{\partial^2 c_t}{\partial (e^{p_t})^2}(-u''(c_t)) - \big(\frac{\partial c_t}{\partial e^{p_t}}\big)^2 u'''(c_t) &= E_t[\Bigg(\frac{\partial^2 a_{t+1}}{\partial (e^{p_t})^2}\frac{\partial c_{t+1}}{\partial a_{t+1}} + \big(\frac{\partial a_{t+1}}{\partial e^{p_t}}\big)^2\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2} \\
&+ \frac{\partial a_{t+1}}{\partial e^{p_t}}\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_{t+1}\partial e^{p_{t+1}}} + \frac{\partial^2 e^{p_{t+1}}}{\partial (e^{p_t})^2}\frac{\partial c_{t+1}}{\partial e^{p_{t+1}}} + \frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial a_{t+1}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_{t+1}\partial e^{p_{t+1}}} \\
&+ \big(\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\big)^2\frac{\partial^2 c_{t+1}}{\partial (e^{p_{t+1}})^2}\Bigg)(-u''(c_{t+1}))] - E_t[\big(\frac{\partial c_{t+1}}{\partial e^{p_t}}\big)^2 u'''(c_{t+1})]
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\frac{\partial^2 c_t}{\partial (e^{p_t})^2} &= -\frac{\partial^2 c_t}{\partial (e^{p_t})^2}E_t[\frac{\partial c_{t+1}}{\partial a_{t+1}}\frac{-u''(c_{t+1})}{-u''(c_t)}] \\
&+ E_t[\Bigg(\big(\frac{\partial a_{t+1}}{\partial e^{p_t}}\big)^2\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2} + 2\frac{\partial a_{t+1}}{\partial e^{p_t}}\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_{t+1}\partial e^{p_{t+1}}} + \big(\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\big)^2\frac{\partial^2 c_{t+1}}{\partial (e^{p_{t+1}})^2}\Bigg)\frac{-u''(c_{t+1})}{-u''(c_t)}] \\
&- \frac{u'''(c_t)}{-u''(c_t)}\Bigg(E_t[\big(\frac{\partial c_{t+1}}{\partial e^{p_t}}\big)^2\frac{u'''(c_{t+1})}{u'''(c_t)}] - E_t[\frac{\partial c_{t+1}}{\partial e^{p_t}}\frac{-u''(c_{t+1})}{-u''(c_t)}]^2\Bigg)
\end{aligned} \tag{3.5}$$

From Lemma 4 in Commault (2024), when the ratios of temperance over prudence and prudence over risk-aversion are non-increasing, then $1 \geq E_t[\frac{(-u''(c_{t+1})^2/u'''(c_{t+1})}{(-u''(c_t)^2/u'''(c_t)}]$

$$\begin{aligned}
&\frac{\partial^2 c_t}{\partial (e^{p_t})^2}\Bigg(1 + E_t[\frac{\partial c_{t+1}}{\partial a_{t+1}}\frac{-u''(c_{t+1})}{-u''(c_t)}]\Bigg) \\
&\leq E_t[\Bigg(\big(\frac{\partial a_{t+1}}{\partial e^{p_t}}\big)^2\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2} + 2\frac{\partial a_{t+1}}{\partial e^{p_t}}\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_{t+1}\partial e^{p_{t+1}}} + \big(\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\big)^2\frac{\partial^2 c_{t+1}}{\partial (e^{p_{t+1}})^2}\Bigg)\frac{-u''(c_{t+1})}{-u''(c_t)}] \\
&- \frac{u'''(c_t)}{-u''(c_t)}\Bigg(E_t[\big(\frac{\partial c_{t+1}}{\partial e^{p_t}}\big)^2\frac{u'''(c_{t+1})}{u'''(c_t)}]E_t[\frac{(-u''(c_{t+1})^2/u'''(c_{t+1})}{(-u''(c_t)^2/u'''(c_t)}] - E_t[\frac{\partial c_{t+1}}{\partial e^{p_t}}\frac{-u''(c_{t+1})}{-u''(c_t)}]^2\Bigg)
\end{aligned} \tag{3.6}$$

Using the more compact notations $P_{t+1} = \frac{\partial c_{t+1}}{\partial e^{p_t}} \frac{-u''(c_{t+1})}{-u''(c_t)}$ and $U_{t+1} = \frac{u'''(c_{t+1})/(-u''(c_{t+1}))^2}{u'''(c_t)/(-u''(c_t))^2}$, I have

$$\begin{aligned}
&\frac{\partial^2 c_t}{\partial (e^{p_t})^2}\left(1 + E_t[\frac{\partial c_{t+1}}{\partial a_{t+1}} \frac{-u''(c_{t+1})}{-u''(c_t)}]\right) \\
&\leq E_t[\underbrace{\left((\frac{\partial a_{t+1}}{\partial e^{p_t}})^2 \frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2} + 2\frac{\partial a_{t+1}}{\partial e^{p_t}} \frac{\partial e^{p_{t+1}}}{\partial e^{p_t}} \frac{\partial^2 c_{t+1}}{\partial a_{t+1} \partial e^{p_{t+1}}} + (\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}})^2 \frac{\partial^2 c_{t+1}}{\partial (e^{p_{t+1}})^2} \right)}_{<0 \text{ from the Lemma}} \frac{-u''(c_{t+1})}{-u''(c_t)}] \\
&\quad - \frac{u'''(c_t)}{-u''(c_t)} \underbrace{\left(E_t[P_{t+1}^2 U_{t+1}] E_t[U_{t+1}^{-1}] - E_t[P_{t+1}]^2 \right)}_{>0 \text{ with Cauchy-Schwartz}} \Bigg) < 0
\end{aligned} \tag{3.7}$$

Because if the Theorem holds true not strictly at $t+1$, then it holds true strictly at t, then holds true strictly at any $t \leq T-1$.

4 Conclusion

Consumption is concave in permanent earnings in the standard income-fluctuation model, for a large range of utility functions. This extends the human capital the influential concavity result of Carroll and Kimball (1996) for risk-free accumulated capital.